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Matrix representation of the generators of symplectic algebras: I. The case of sp(4, *R*)

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Abstract. Because of their many physical applications, there has been considerable interest lately in the basis for irreducible representations (irreps) of symplectic groups and in the matrix elements of the generators of these groups with respect to this basis. In the present paper we carry out this programme for the irreps in the positive discrete series of sp(4, *R*) by building up this basis from powers of the raising generators applied to the lowest weight state. By using the Dyson boson realisation of the generators, their matrix representation can be obtained by direct differentiation, while the overlap of the states of the basis can be determined with the help of coherent states. The extension of the analysis to symplectic groups of arbitrary dimension is straightforward and will be implemented explicitly for sp(6, *R*) in a later paper.

1. Introduction and summary

In the past few years there has been considerable interest in the basis for irreducible representations (irreps) of symplectic groups (Deenen and Quesne 1982, 1984a, b, Kramer 1982, Kramer *et al* 1984, 1985, Rowe *et al* 1984, Castaños *et al* 1985a, b, Moshinsky 1985, Hecht 1985) and the matrix elements of the generators of the groups with respect to this basis, particularly because of their applications to collective motions in nuclei (Dzublik *et al* 1972, Filippov and Ovcharenko 1979, Filippov *et al* 1980, 1981, Vanagas 1977, 1980, Rosensteel and Rowe 1980, Park *et al* 1984, Draayer *et al* 1984, Moshinsky 1984b, Chacón *et al* 1984, Castaños *et al* 1984, Suzuki and Hecht 1986). The authors and their collaborators have made contributions to this programme (Castaños *et al* 1985a, b, 1986, Moshinsky 1985) emphasising the construction of the basis in terms of the coordinates or, equivalently, the creation operators associated with the many-body systems. A more abstract procedure working in the enveloping algebra of the Lie algebra in question is well known (Jacobson 1962) and has been applied very explicitly by Gruber and Klymik (1984) for Su(2) and other groups. In the present paper we wish to follow this approach for the discussion of the basis of irreps, and the matrix elements of the generators with respect to this basis, for the case of sp(4, *R*) to extend it later to the physically interesting case of sp(6, *R*).

The analysis in this paper will proceed as follows. In § 2 we will introduce for su(2) the concept of coherent states and boson realisations, which will allow us to obtain, in this simple case, the well known matrix representation of the generators of

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the su(2) Lie algebra, in a way that will parallel the approach we follow later for the symplectic case. In § 3 we introduce the generators of sp(4, R) in vector form with spherical rather than cartesian components, discuss their commutation relations and then construct the states in the enveloping algebra that correspond to the irreps of the positive discrete series. In § 4 we introduce the Dyson boson realisation (Castaños *et al* 1985a, b, 1986) for the generators, which allows us to apply them to the basis by simple differentiation and thus determine the matrix representation of the generators for a given irrep of sp(4, R). In § 5 we note that, as our basis is not orthonormal, we require the overlap of its states, which can be obtained with the help of the coherent states associated with the problem. We furthermore indicate that the combination of the matrix representation discussed above and the overlaps determined in this section allow us to obtain the matrix elements of operator functions of the generators between the bras and kets that are part of the basis associated with a definite irrep of sp(4, R). In § 6 we discuss the Casimir operators of the different subalgebras of sp(4, R) that are of physical interest and obtain their matrix representation, which allows us to determine the eigenvalues of any Hamiltonian given by a linear combination of these Casimir operators as well as of other elements in the enveloping algebra of sp(4, R). Finally in the concluding section we indicate how the preceding analysis can be generalised to sp($2d$, R), where d is any integer, and its relevance for collective motions in nuclei which correspond to $d=3$.

2. Applications to su(2)

Before proceeding with the main objective of this paper, we shall illustrate our procedures in the simple case of su(2) where, in particular, we shall show how the use of coherent states and of the boson realisation of this algebra allow us to implement our program.

We designate the generators of su(2) by the operators

$$S_1 = -\frac{1}{\sqrt{2}}(S_x + \mathrm{i}S_y) \qquad S_0 = S_z \qquad S_{-1} = \frac{1}{\sqrt{2}}(S_x - \mathrm{i}S_y) \tag{2.1}$$

satisfying the commutation relations

$$[S_0, S_{\pm 1}] = \pm S_{\pm 1} \qquad [S_{-1}, S_1] = S_0. \tag{2.2}$$

In the usual basis, to be denoted by $|s\sigma\rangle$, the matrix representation of these generators has the well known form

$$\langle s\sigma'|S_{\pm 1}|s\sigma\rangle = \mp(1/\sqrt{2})[(s\mp\sigma)(s\pm\sigma+1)]^{1/2}\delta_{\sigma',\sigma\pm 1} \tag{2.3a}$$

$$\langle s\sigma'|S_0|s\sigma\rangle = \sigma\delta_{\sigma',\sigma}. \tag{2.3b}$$

We also have the possibility of introducing the basis

$$|s\sigma) \equiv (S_1)^{s+\sigma}|s, -s\rangle \tag{2.4}$$

where $|s, -s\rangle$ is the lowest weight state

$$S_{-1}|s, -s\rangle = 0 \qquad S_0|s, -s\rangle = -s|s, -s\rangle. \tag{2.5a, b}$$

Applying then the generators $S_{\pm 1}$, S_0 to the round kets of (2.4) we obtain

$$S_1|s\sigma) = S_1^{s+\sigma+1}|s, -s\rangle = |s\sigma+1) \tag{2.6a}$$

$$S_0|s\sigma)=[S_0, S_1^{s+\sigma}]|s,-s\rangle+S_1^{s+\sigma}S_0|s,-s\rangle=\sigma|s\sigma) \tag{2.6b}$$

$$S_{-1}|s\sigma)=\sum_{r=0}^{s+\sigma-1} S_1^r[S_{-1}, S_1]S_1^{s+\sigma-r-1}|s,-s\rangle=-\tfrac{1}{2}(s+\sigma)(s-\sigma+1)|s,\sigma-1) \tag{2.6c}$$

thus obtaining a different matrix representation $\|\mathfrak{M}^q_{\sigma'\sigma}\|$ of S_q, $q=1, 0, -1$, through the relation

$$S_q|s\sigma)=\sum_{\sigma'}|s\sigma')\mathfrak{M}^q_{\sigma'\sigma} \tag{2.7}$$

where the explicit form of $\mathfrak{M}^q_{\sigma'\sigma}$; σ', $\sigma=s, s-1,\ldots,-s$, $q=1, 0, -1$, is obtained from (2.6). It is easy to check with the help of (2.6) that the Casimir operator

$$S^2=-S_1S_{-1}-S_{-1}S_1+S_0^2 \tag{2.8}$$

when applied to the state $|s\sigma)$ gives

$$S^2|s\sigma)=s(s+1)|s\sigma) \tag{2.9}$$

as it should.

The basis (2.4) is a very simple one and can be generalised immediately to some of the irreps of other groups, but it has the disadvantage that the overlap of its states

$$(s\sigma'|s\sigma) \tag{2.10}$$

while zero for $\sigma'\neq\sigma$, as σ, σ' are eigenvalues of the Hermitian operator S_0, is not one for $\sigma'=\sigma$. The value of $(s\sigma|s\sigma)$ can be easily obtained from the definition (2.4) and the commutation relations (2.2), but we prefer to use coherent states of su(2) for this purpose, which will provide a technique to be used in the overlap of states associated with irreps of symplectic algebras.

Let us define the coherent states (Kramer and Saraceno 1981)

$$|y\rangle=\exp(\bar{y}S_1)|s,-s\rangle \qquad \langle y'|=\langle s,-s|\exp(-y'S_{-1}) \tag{2.11a, b}$$

where y, y' are complex numbers and $\bar{y}$ is the conjugate of y. Their overlap is then given by

$$\langle y'|y\rangle=\langle s,-s|\exp(-y'S_{-1})\exp(\bar{y}S_1)|s,-s\rangle$$
$$=\sum_{\sigma,\sigma'=-s}^{s}[(s+\sigma)!(s+\sigma')!]^{-1}\bar{y}^{s+\sigma}y'^{s+\sigma'}\langle s,-s|(-S_{-1})^{s+\sigma'}(S_1)^{s+\sigma}|s,-s\rangle \tag{2.12}$$

and thus, if we have an independent way of obtaining $\langle y'|y\rangle$ as a function of y', $\bar{y}$, we can obtain, from its development in series, the overlap (2.10).

As $\exp(-y'S_{-1})\exp(\bar{y}S_1)$ is a finite su(2) transformation, we can express it in a 'time ordered' fashion, i.e.

$$\exp(-y'S_{-1})\exp(\bar{y}S_1)=\exp(aS_1)\exp(bS_0)\exp(cS_{-1}) \tag{2.13}$$

where to find a, b, c it is sufficient to express, on both sides of (2.13), the operators S_q in terms of Pauli matrices, i.e.

$$S_1=-\frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \qquad S_0=\frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \qquad S_{-1}=\frac{1}{\sqrt{2}}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}. \tag{2.14}$$

We then have that

$$\exp(-y'S_{-1})\exp(\bar{y}S_1)=\left[I-\frac{y'}{\sqrt{2}}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right]\left[I-\frac{\bar{y}}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right]=\begin{bmatrix}1 & -\bar{y}/\sqrt{2}\\ y'/\sqrt{2} & 1+(y'\bar{y})/2\end{bmatrix} \tag{2.15}$$

while

$$\exp(aS_1)\exp(bS_0)\exp(cS_{-1})=\begin{bmatrix} e^{b/2}-(ac/2)\,e^{-b/2} & -a/\sqrt{2}\,e^{-b/2} \\ c/\sqrt{2}\,e^{-b/2} & e^{-b/2}\end{bmatrix} \tag{2.16}$$

which gives us

$$e^{-b}=[1+(y'\bar{y}/2)]^2. \tag{2.17}$$

From (2.12), (2.13) and (2.5b) we then have that

$$\begin{aligned}\langle y'|y\rangle &= \langle s,-s|\exp(aS_1)\exp(bS_0)\exp(cS_{-1})|s,-s\rangle \\ &= \langle s,-s|\exp(bS_0)|s,-s\rangle=[1+(y'\bar{y}/2)]^{2s} \\ &= \sum_{\sigma=-s}^{s}\frac{(2s)!}{(s+\sigma)!(s-\sigma)!}\left(\frac{1}{2}\right)^{s+\sigma} y'^{s+\sigma}\bar{y}^{s+\sigma}.\end{aligned} \tag{2.18}$$

From this expression and (2.12) we than conclude that the overlap becomes

$$(s\sigma'|s\sigma)=\frac{(2s)!(s+\sigma)!}{2^{s+\sigma}(s-\sigma)!}\,\delta_{\sigma'\sigma}. \tag{2.19}$$

As a final point concerning su(2) we consider its realisation in terms of creation α and annihilation $\bar{\alpha}$ operators satisfying the commutation rule

$$[\bar{\alpha},\alpha]=1. \tag{2.20}$$

We easily check with the help of (2.20) that (Kramer and Saraceno 1981)

$$S_1=-\alpha(\alpha\bar{\alpha}-2s) \tag{2.21a}$$

$$S_0=\alpha\bar{\alpha}-s \tag{2.21b}$$

$$S_{-1}=-\tfrac{1}{2}\bar{\alpha} \tag{2.21c}$$

satisfy the commutation rules (2.2), and besides that, if we define the state

$$|s\sigma\}=\frac{1}{(s-\sigma)!}\,\alpha^{s+\sigma}|0\} \qquad \bar{\alpha}|0\}=0 \tag{2.22a, b}$$

the application of the S_q, $q=1, 0, -1$, of (2.21) to it gives precisely the expressions in (2.6). In this boson realisation the generators (2.21) can be interpreted as differential operators from the fact that $\bar{\alpha}=\partial/\partial\alpha$ will satisfy (2.20). Thus the application of S_q to the states (2.22) is simplified, as it only implies a differentiation.

We now pass to sp(4, R) where several of the previous considerations will be applied.

3. Generators of sp(4, *R*) and basis for the irreps in the enveloping algebra

In previous papers we gave the generators of sp(4, R) in vector form with cartesian components. In this paper we prefer to express the vectors in spherical components $q=1, 0, -1$, as in (2.1). We thus have the ten generators as (Castaños *et al* 1985a, b, 1986)

$$\mathcal{N},\ B_q^{\dagger},\ J_q,\ B_q \tag{3.1}$$

and from their commutation relations in cartesian components we obtain

$$[\mathcal{N}, B_q^{\dagger}]=B_q^{\dagger}, \qquad [\mathcal{N}, B_q]=-B_q \qquad [\mathcal{N}, J_q]=0 \qquad [B_q^{\dagger}, B_{q'}^{\dagger}]=[B_q, B_{q'}]=0 \tag{3.2a}$$

while the rest of the commutation relations are given in the following table:

	$B_1^\dagger$	$B_0^\dagger$	$B_{-1}^\dagger$	J_1	J_0	J_{-1}
B_1	0	$2J_1$	$-2(\mathcal{N}-J_0)$	0	$-B_1$	$-B_0$
B_0	$-2J_1$	$2\mathcal{N}$	$2J_{-1}$	B_1	0	$-B_{-1}$
B_{-1}	$-2(\mathcal{N}+J_0)$	$-2J_{-1}$	0	B_0	B_{-1}	0
J_1	0	$-B_1^\dagger$	$-B_0^\dagger$	0	$-J_1$	$-J_0$
J_0	$B_1^\dagger$	0	$-B_{-1}^\dagger$	J_1	0	$-J_{-1}$
J_{-1}	$B_0^\dagger$	$B_{-1}^\dagger$	0	J_0	J_{-1}	0

(3.2*b*)

where the term appearing in the body of the table is the commutator of the one appearing on the right-hand side in the corresponding row and the one above in the corresponding column, e.g. $[B_{-1}, B_0^\dagger] = -2J_{-1}$.

Note that the J_q, $q = 1, 0, -1$, are the generators in spherical components of the su(2) subalgebra of sp (4, R) with the standard angular momentum properties while, with respect to J_q, the $B_q^\dagger$, $B_{q'}$, behave as three-dimensional vectors.

The set of generators (3.1) can be divided into three subsets of raising, weight and lowering type, which are separated by semicolons (Castaños *et al* 1985a, b, 1986):

$$B_q^\dagger, J_1; \mathcal{N}, J_0; B_q, J_{-1}. \tag{3.3}$$

The lowest weight state, which we designate by $|w, s, -s\rangle$ can now be characterised by

$$B_q|w, s, -s\rangle = 0 \qquad q = 1, 0, -1 \tag{3.4a}$$

$$J_{-1}|w, s, -s\rangle = 0 \tag{3.4b}$$

$$\mathcal{N}|w, s, -s\rangle = w|w, s, -s\rangle \tag{3.4c}$$

$$J_0|w, s, -s\rangle = -s|w, s, -s\rangle \tag{3.4d}$$

where w, s are integer or semi-integer numbers. This state is the lowest weight one of the irrep $[w-s, w+s]$ in the discrete positive series. Clearly

$$J^2|w, s, -s\rangle = [-2J_1J_{-1} + J_0(J_0 - 1)]|w, s, -s\rangle = s(s+1)|w, s, -s\rangle \tag{3.5}$$

and this is the reason for the notation in the ket as it is characterised by the eigenvalues w, $s(s+1)$, $-s$ of $\mathcal{N}$, J^2, J_0.

The analysis of Gruber and Klymik (1984), as well as a previous discussion (Castaños *et al* 1986), indicates that the full basis for the irrep $[w-s, w+s]$ of sp(4, R) is given by applying powers of the raising generators $B_1^\dagger$, $B_0^\dagger$, $B_{-1}^\dagger$, J_1, i.e. elements of the enveloping algebra, to the lowest weight state $|w, s, -s\rangle$. Thus we can characterise this basis by

$$|N, M, \mu, \sigma\rangle = (B_1^\dagger)^{(N+M-\mu-\sigma)/2}(B_0^\dagger)^{\mu}(B_{-1}^\dagger)^{(N-M-\mu+\sigma)/2}(J_1)^{s+\sigma}|w, s, -s\rangle \tag{3.6}$$

where the choice of exponents guarantees, from the commutation relation (3.2), that the ket (3.6) is an eigenstate of the weight generators $\mathcal{N}$, J_0, i.e.

$$\mathcal{N}|N, M, \mu, \sigma\rangle = (N+w)|N, M, \mu, \sigma\rangle \qquad J_0|N, M, \mu, \sigma\rangle = M|N, M, \mu, \sigma\rangle. \tag{3.7a, b}$$

Note that kets depend on (w, s) but as the irrep of sp(4, R) is kept fixed we do not include it.

By applying the generators (3.1) of sp(4, R) to the states (3.6), with the help of the commutation relation (3.2), we can obtain, in analogy with the analysis (2.6) for su(2), the matrix representation of these generators for the irrep $[w-s, w+s]$ of sp(4, R). The use of the commutation relation is rather cumbersome and thus in the next section we shall discuss a Dyson-type boson realisation of sp(4, R) that will allow us to obtain this matrix representation by simple differentiation, in analogy with the one carried out explicitly for su(2) at the end of § 2.

4. The Dyson boson realisation and the matrix representation of the generators of sp(4, *R*)

In previous papers (Castaños *et al* 1985a, b, 1986) we have shown how the generators of sp(4, R) can be expressed in terms of those of the direct sum of $\mathscr{W}(3)$ and su(2), where the former is a Weyl Lie algebra in three dimensions whose generators are the creation β_q and annihilation operators $\bar{\beta}^q=(-1)^q\bar{\beta}_{-q}$, $q=1,0,-1$, satisfying

$$[\beta_q,\beta_{q'}]=[\bar{\beta}_q,\bar{\beta}_{q'}]=0 \qquad [\bar{\beta}^q,\beta_{q'}]=\delta^q_{q'} \tag{4.1a, b}$$

while the latter has as generators the spin operators S_q satisfying

$$[S_0,S_{\pm1}]=\pm S_{\pm1} \qquad [S_{-1},S_1]=S_0. \tag{4.1c, d}$$

Furthermore $\mathscr{W}(3)$ and su(2) are independent so that

$$[\beta_q,S_q]=[\bar{\beta}_q,S_{q'}]=0. \tag{4.1e}$$

Note that here we give all generators of $\mathscr{W}(3)\oplus$su(2) in spherical rather than cartesian components and use the notation β_q, $\bar{\beta}_q$, S_q; $q=1,0,-1$, instead of the β_i^+, β_i, S_i; $i=1,2,3$, of Castaños *et al* (1985a).

In vector notation the realisation of the ten generators (3.1) of sp(4, R) in terms of $\boldsymbol{\beta}$, $\bar{\boldsymbol{\beta}}$, $\boldsymbol{S}$ and the eigenvalue w of (3.4c) is given by Castaños *et al* (1985a):

$$\boldsymbol{B}^+=\boldsymbol{\beta} \qquad \boldsymbol{J}=\boldsymbol{L}+\boldsymbol{S} \qquad \mathcal{N}=\mathfrak{N}+w \tag{4.2a, b, c}$$

$$\boldsymbol{B}=-\boldsymbol{\beta}(\bar{\boldsymbol{\beta}}\cdot\bar{\boldsymbol{\beta}})+(2\mathfrak{N}+2w)\bar{\boldsymbol{\beta}}-2\mathrm{i}(\bar{\boldsymbol{\beta}}\times\boldsymbol{S}) \tag{4.2d}$$

where

$$\boldsymbol{L}=-\mathrm{i}(\boldsymbol{\beta}\times\bar{\boldsymbol{\beta}}) \qquad \mathfrak{N}=\boldsymbol{\beta}\cdot\bar{\boldsymbol{\beta}}. \tag{4.3a, b}$$

We can immediately check from (4.1) that $\mathcal{N}$, B_q^+, J_q, B_q of (4.2) satisfy the commutation rules (3.2). Note though that from the Hermitian properties

$$(\boldsymbol{B}^+)^+=\boldsymbol{B} \qquad \boldsymbol{J}^+=\boldsymbol{J} \qquad \mathcal{N}^+=\mathcal{N} \tag{4.4a, b, c}$$

of the generators of sp(4, R) we conclude

$$\boldsymbol{\beta}^+\neq\bar{\boldsymbol{\beta}} \tag{4.5}$$

and thus we are dealing with what is known as a Dyson-type boson realisation of sp(4, R) instead of a Holstein–Primakoff one.

We can furthermore express the S_q, $q=1$, 0, -1, in the boson realisation (2.21), which is again of the Dyson type, and thus we see that we can write $\mathcal{N}$, B_q^+, J_q, B_q in terms of the creation operators β_q, α and the annihilation ones $\bar{\beta}^q$, $\bar{\alpha}$ which, besides the commutation rules (4.1a, b), satisfy also

$$[\beta_q,\alpha]=[\bar{\beta}_q,\alpha]=[\bar{\beta}_q,\bar{\alpha}]=[\beta_q,\bar{\alpha}]=0, \qquad [\bar{\alpha},\alpha]=1. \tag{4.6a, b}$$

We now define the boson states

$$|N, M, \mu, \sigma\} = (\beta_1)^{(N+M-\mu-\sigma)/2}(\beta_0)^{\mu}(\beta_{-1})^{(N-M-\mu+\sigma)/2}\alpha^{s+\sigma}/(s-\sigma)!|0\} \tag{4.7}$$

where the boson vacuum $|0\}$ has the property

$$\bar{\beta}_q|0\} = 0 \qquad q = 1, 0, -1 \qquad \bar{\alpha}|0\} = 0. \tag{4.8a, b}$$

From (2.22) we then immediately conclude that applying the operators (4.2), in which S_q is replaced by (2.21), to the states (4.7) gives the same result as applying the generators (3.1) of sp(4, R) directly to the states (3.6) in the enveloping algebra and using the commutation relations to obtain the matrix representation of the generators.

The fact that β_q, α; $\bar{\beta}^q$, $\bar{\alpha}$ satisfy the commutation relations (4.1a, b) and (4.6a, b) make it much easier to apply (4.2), in which we replace S_q by (2.21), to the states (4.7), as we can interpret

$$\bar{\beta}^q = \partial/\partial\beta_q \qquad q = 1, 0, -1 \qquad \bar{\alpha} = \partial/\partial\alpha. \tag{4.9a, b}$$

Thus the expressions (4.2) for the generators of sp(4, R) become simple multiplicative or differential operators in the variables β_q, α that we have to apply to the products of powers of β_1, β_0, β_{-1}, α appearing in (4.7), from which we obtain immediately the matrix representation of the generators of sp(4, R). Translating these results to $\mathcal{N}$, $B_q^\dagger$, J_q, B_q of (3.1), being applied to the states (3.6), we then obtain

$$\mathcal{N}|N, M, \mu, \sigma\rangle = (N+w)|N, M, \mu, \sigma\rangle \qquad J_0|N, M, \mu, \sigma\rangle = M|N, M, \mu, \sigma\rangle \tag{4.10a, b}$$

as also follows from (3.7), plus the following expressions:

$$J_1|N, M, \mu, \sigma\rangle = -\mu|N, M+1, \mu-1, \sigma\rangle + |N, M+1, \mu, \sigma+1\rangle - \tfrac{1}{2}(N-M+\sigma-\mu)|N, M+1, \mu+1, \sigma\rangle \tag{4.10c}$$

$$J_{-1}|N, M, \mu, \sigma\rangle = \tfrac{1}{2}(N+M-\mu-\sigma)||N, M-1, \mu+1, \sigma\rangle + \mu|N, M-1, \mu-1, \sigma\rangle - \tfrac{1}{2}(s+\sigma)(s-\sigma+1)|N, M-1, \mu, \sigma-1\rangle \tag{4.10d}$$

$$B_1^\dagger|N, M, \mu, \sigma\rangle = |N+1, M+1, \mu, \sigma\rangle \tag{4.10e}$$

$$B_0^\dagger|N, M, \mu, \sigma\rangle = |N+1, M, \mu+1, \sigma\rangle \tag{4.10f}$$

$$B_{-1}^\dagger|N, M, \mu, \sigma\rangle = |N+1, M-1, \mu, \sigma\rangle \tag{4.10g}$$

$$B_1|N, M, \mu, \sigma\rangle = -\tfrac{1}{2}(N+2w-M-\sigma+\mu-2)(N-M+\sigma-\mu)|N-1, M+1, \mu, \sigma\rangle - \mu(\mu-1)|N-1, M+1, \mu-2, \sigma\rangle + 2\mu|N-1, M+1, \mu-1, \sigma+1\rangle \tag{4.10h}$$

$$B_0|N, M, \mu, \sigma\rangle = \tfrac{1}{2}(N+M-\sigma-\mu)(N-M+\sigma-\mu)|N-1, M, \mu+1, \sigma\rangle + \mu(2N+2w-\mu-1,)|N-1, M, \mu-1, \sigma\rangle - (N+M-\sigma-\mu)|N-1, M, \mu, \sigma+1\rangle - \tfrac{1}{2}(s+\sigma)(s-\sigma+1)(N-M+\sigma-\mu)|N-1, M, \mu, \sigma-1\rangle \tag{4.10i}$$

$$B_{-1}|N, M, \mu, \sigma\rangle = -\tfrac{1}{2}(N+2w+M+\sigma+\mu-2)(N+M-\mu-\sigma)|N-1, M-1, \mu, \sigma\rangle - \mu(\mu-1)|N-1, M-1, \mu-2, \sigma\rangle + \mu(s+\sigma)(s-\sigma+1)|N-1, M-1, \mu-1, \sigma-1\rangle. \tag{4.10j}$$

If we express generically by X one of the generators of sp(4, R) we will have

$$X|N, M, \mu, \sigma\rangle = \sum_{N'M'\mu'\sigma'} |N', M', \mu', \sigma'\rangle \mathfrak{M}^{X}_{N'M'\mu'\sigma', NM\mu\sigma} \tag{4.11}$$

where

$$\mathbf{X} \equiv \|\mathfrak{M}^{X}_{N'M'\mu'\sigma', NM\mu\sigma}\| \tag{4.12}$$

is the matrix representation of X in the enveloping algebra basis (3.6), whose elements can be obtained explicitly by comparing (4.11) with (4.10).

The basis (3.6) is very simple if we wish to determine the matrix representation, but unfortunately it is not an orthonormal one. In the next section we shall use coherent states of sp(4, R) to find the overlaps of the kets (3.6) and thus complete the procedure for finding representations also in orthonormal basis, for which these overlaps need to be diagonalised.

5. Coherent states and the determination of overlaps

Our objective now is to determine the overlaps of the states (3.6) in which for compactness we write

$$\tfrac{1}{2}(N+M-\mu-\sigma) \equiv \lambda \qquad \tfrac{1}{2}(N-M-\mu+\sigma) \equiv \nu. \tag{5.1}$$

We thus have

$$\begin{aligned}\langle \lambda'\mu'\nu'\sigma'|\lambda\mu\nu\sigma\rangle &= \langle w, s, -s|(J^1)^{s+\sigma'}(B^1)^{\lambda'}(B^0)^{\mu'}(B^{-1})^{\nu'} \\ &\quad \times (B_1^\dagger)^{\lambda}(B_0^\dagger)^{\mu}(B_{-1}^\dagger)^{\nu}(J_1)^{s+\sigma}|w, s, -s\rangle \end{aligned}\tag{5.2}$$

where $B^q = (-1)^q B_{-q}$, $q = 1, 0, -1$, $J^1 = -J_{-1}$. Note that λ', ν' in (5.2) are given by the relations (5.1) in which we replace μ, σ by μ', σ' but keep the same N, M as they are eigenvalues of the Hermitian operators $\mathcal{N} - w$, J_0.

We now carry out our analysis in exactly the same fashion as in the discussion of the overlaps for the states $|s\sigma\rangle$ of su(2), given between equations (2.11) and (2.19) of § 2. We introduce the coherent states of sp(4, R) by the definitions

$$|yz\rangle = \exp(\bar{z}^q B_q^\dagger)\exp(\bar{y}J_1)|w, s, -s\rangle \tag{5.3a}$$

$$\langle y'z'| = \langle w, s, -s|\exp(y'J^1)\exp(z'_q B^q) \tag{5.3b}$$

where y', y are complex numbers and z'_q, z_q the spherical components of complex vectors. The bar above the letters means complex conjugate and the repeated index implies sums over its values $q = 1, 0, -1$, while the raising or lowering of the indices follows the rules indicated after (5.2).

From (3.2a) the $B_1^\dagger$, $B_0^\dagger$, $B_{-1}^\dagger$ commute among themselves as also happens for B^1, B^0, B^{-1}. Thus taking the scalar product of the coherent states in (5.3) we can write

$$\begin{aligned}\langle y'z'|yz\rangle = \sum_{\lambda'\mu'\nu'\sigma'}\sum_{\lambda\mu\nu\sigma}\bigg(&\frac{y'^{s+\sigma'}}{(s+\sigma')!}\frac{(z'_1)^{\lambda'}}{\lambda'!}\frac{(z'_0)^{\mu'}}{\mu'!} \\ &\times\frac{(z'_{-1})^{\nu'}}{\nu'!}\frac{\bar{y}^{s+\sigma}}{(s+\sigma)!}\frac{(\bar{z}^1)^{\lambda}}{\lambda!}\frac{(\bar{z}^0)^{\mu}}{\mu!}\frac{(\bar{z}^{-1})^{\nu}}{\nu!} \\ &\times\langle w, s, -s|(J^1)^{s+\sigma'}(B^1)^{\lambda'}(B^0)^{\mu'}(B^{-1})^{\nu'}(B_1^\dagger)^{\lambda}(B_0^\dagger)^{\mu} \\ &\times(B_{-1}^\dagger)^{\nu}(J_1)^{s+\sigma}|w, s, -s\rangle\bigg).\end{aligned}\tag{5.4}$$

If we have an independent way of determining $\langle y'z'|yz\rangle$ we see that by expanding it in terms of powers of y', z'_q, $\bar{y}$, $\bar{z}^q$, $q=1, 0, -1$, we can immediately obtain the overlaps (5.2).

Fortunately the explicit expression of $\langle y'z'|yz\rangle$ was obtained by the authors in collaboration with Kramer (Castaños *et al* 1986). They used the defining representation of sp(4, R) in terms of 4×4 matrices (Gilmore 1974), in the same way as the Pauli spin matrices were used for su(2) in equations (2.14)-(2.18), i.e. to express the exponentials of the generators appearing in $\langle y'z'|yz\rangle$ in a 'time ordered' fashion, first exponentials in $B_q^\dagger$, then in J_q and finally in B^q. From the definition (3.4) of the lowest weight state $|w, s, -s\rangle$ we have that

$$\exp(c_q B^q)|w, s, -s\rangle = |w, s, -s\rangle \qquad \langle w, s, -s|\exp(a^q B_q^\dagger) = \langle w, s, -s| \tag{5.5a, b}$$

and thus we obtain

$$\langle y'z'|yz\rangle = \langle w, s, -s|\exp(b^q J_q)\exp(\mathrm{d}\mathcal{N})|w, s, -s\rangle \tag{5.6}$$

where a^q, b^q, c_q, d are definite functions of y', $\bar{y}$, z'_q, $\bar{z}^q$, in the same way as, for the case of su(2), the a, b, c appearing in (2.16) were definite functions of y', $\bar{y}$ of (2.15).

The expression (5.6) is a finite representation of a group element of U(2) characterised by the parameters b^q, d and whose generators are J_q, $\mathcal{N}$. This representation has been evaluated by Louck (1970) and using it in our paper with Kramer we arrived at the expression (Castaños *et al* 1986)

$$\langle y'z'|yz\rangle = [\mathcal{M}(\bar{y}\bar{z}; y'z')]^{2s}[\Delta(\bar{z}, z')]^{-(w+s)} \tag{5.7}$$

where

$$\Delta = 1 - 2z'\cdot\bar{z} + (z'\cdot z')(\bar{z}\cdot\bar{z}) \tag{5.8a}$$

$$\mathcal{M} = h + ky' + l\bar{y} + my'\bar{y} \tag{5.8b}$$

with

$$h = 1 - 2z'_1\bar{z}^1 - z'_0\bar{z}^0 \tag{5.9a}$$

$$k = -z'_0\bar{z}^1 - z'_{-1}\bar{z}^0 \tag{5.9b}$$

$$l = -z'_1\bar{z}^0 - z'_0\bar{z}^{-1} \tag{5.9c}$$

$$m = \tfrac{1}{2}(1 - 2z'_{-1}\bar{z}^{-1} - z'_0\bar{z}^0). \tag{5.9d}$$

We now have to expand $\langle y'z'|yz\rangle$ in terms of powers of y', z'_q, $\bar{y}$, $\bar{z}^q$ and compare with (5.4) to obtain the explicit expression for the overlap (5.2). The detailed analysis is given in the appendix and we just state here the result

$$\begin{aligned}\langle\lambda', \mu', \nu', \sigma'|\lambda, \mu, \nu, \sigma\rangle &= \lambda'!\mu'!\nu'!(s+\sigma)!(s+\sigma')!\lambda!\nu!\mu!\\ &\quad\times\sum_{\gamma,\beta,\alpha}\sum_{\tau,\tau'} B_{\tau\tau'}(\gamma, \beta, \alpha, w, s)(\gamma\tau\alpha|\gamma\beta\alpha)(\gamma\tau'\alpha|\gamma\beta\alpha)\\ &\quad\times C[\lambda'-(\gamma-\tau+\alpha)/2, \mu'-\tau, \nu'-(\gamma-\tau-\alpha)/2, \sigma';\\ &\quad\times\lambda-(\gamma-\tau'-\alpha)/2, \mu-\tau', \nu-(\gamma-\tau'+\alpha)/2, \sigma]\end{aligned} \tag{5.10}$$

where (|) denotes the transformation brackets between the spherical harmonic oscillator states and the cylindrical ones (Chacón and de Llano 1963) given in (A9). The coefficient B is

$$B_{\tau\tau'}(\gamma, \beta, \alpha, w, s) = (-1)^{\gamma-\tau'} \frac{(\gamma+2w+2s-\beta-3)!!}{(2w+2s-2)!!} \frac{(\gamma+2w+2s+\beta-2)!!}{(2w+2s-3)!!}$$
$$\times\{[(\gamma-\tau+\alpha)/2]!\tau![(\gamma-\tau-\alpha)/2]![(\gamma-\tau'+\alpha)/2]!\tau'!$$
$$\times[(\gamma-\tau'-\alpha)/2]!\}^{-1/2} \tag{5.11a}$$

while C corresponds to (A5) with the arguments indicated:

$$C[\lambda'-(\gamma-\tau+\alpha)/2, \mu'-\tau, \nu'-(\gamma-\tau-\alpha)/2, \sigma';$$
$$\lambda-(\gamma-\tau'-\alpha)/2, \mu-\tau', \nu-(\gamma-\tau'+\alpha)/2, \sigma]$$
$$=\sum_r \frac{(-1)^{\lambda'+\mu'+\nu'-\gamma}2^{\lambda'+\nu-\gamma+(\tau+\tau')/2-\alpha-s-\sigma}(2s)!(3s+\sigma-r-\lambda'-\nu+\gamma-(\tau+\tau')/2+\alpha)!}{(2s-\lambda'-\mu'-\nu'+\gamma)!(\mu'+\nu'-\nu-(\tau'+\tau)/2+\alpha+s+\sigma-r)!}$$
$$\times\Bigg(\sum_d [(s+\nu'-\nu-\lambda'+(3\alpha)/2+\gamma/2-\tau'/2-\sigma'-d)!$$
$$\times(\lambda'+\nu-\nu'+s+\sigma-r-(3\alpha)/2-\gamma/2+\tau'/2+d)!$$
$$\times(\nu'-(\gamma-\tau-\alpha)/2-d)!(2s+\sigma+\sigma'-r-\nu'+(\gamma-\tau-\alpha)/2+d)!$$
$$\times(r-s-\gamma-d)!(d)!(\nu-\nu'+(\tau'-\tau)/2-\alpha+d)!$$
$$\times(r-s-\sigma+\nu'-\nu+(\tau-\tau')/2+\alpha-d)!]^{-1}\Bigg). \tag{5.11b}$$

From (5.10) and the relation (5.1) between λ, ν and N, M we then determine the overlap

$$\langle N, M, \mu', \sigma' | N, M, \mu, \sigma\rangle \tag{5.12}$$

and thus if we have some function X of the generators of sp(4, R) we can write from (4.11)

$$\langle N', M', \mu', \sigma'|\mathrm{X}|N, M, \mu, \sigma\rangle = \sum_{\mu''\sigma''} \langle N', M', \mu', \sigma' | N', M', \mu'', \sigma''\rangle \mathfrak{M}^{\mathrm{X}}_{N'M'\mu''\sigma'', NM\mu\sigma}. \tag{5.13}$$

We have thus obtained the matrix element of the operator X between bras and kets that are part of the non-orthonormal basis associated with a definite irrep of sp(4, R).

By determining the orthogonal transformations that diagonalise the matrices whose elements are given by (5.12) with fixed N, M and also using the eigenvalues of these matrices, we can obtain the linear combinations of states (3.6) that are orthonormal, i.e.

$$|N, M, K) = \sum_{\mu,\sigma} A^K_{\mu\sigma}(N, M)|N, M, \mu, \sigma\rangle \tag{5.14}$$

with

$$(N, M, K'|N, M, K) = \delta_{K'K}. \tag{5.15}$$

From (5.14) we then see that the matrix elements of the operator X in this orthornormal basis are given by

$$(N', M', K'|X|N, M, K) = \sum_{\mu',\sigma'} \sum_{\mu,\sigma} [(A^{K'}_{\mu'\sigma'}(N'M')A^{K}_{\mu\sigma}(N, M)$$
$$\times \langle N', M', \mu', \sigma'|X|N, M, \mu, \sigma\rangle)] \tag{5.16}$$

with the matrix element of the right-hand side being given by (5.13).

In the next section we discuss some applications of these results to the matrix representation of Casimir operators of subalgebras of sp(4, R) and of Hamiltonians that can be formed from them and other elements in the enveloping algebra of sp(4, R).

6. Casimir operators of subgroups of sp(4, R) and their matrix representation

We start by looking at the commutation rules (3.2) for the generators of sp(4, R) and consider subsets of these generators that close under commutation, i.e. correspond to subalgebras which we identify, and determine the corresponding Casimir operators.

The first and obvious subalgebra is J_q, $q = 1, 0, -1$, which clearly is of the su(2) type and its Casimir operator is

$$J^2 = -2J_{-1}J_1 + J_0(J_0 + 1). \tag{6.1}$$

We then note from (3.2b) that another subalgebra is given by

$$I_1 = -(1/\sqrt{2})B_0^{\dagger} \qquad I_0 = \mathcal{N} \qquad I_{-1} = (1/\sqrt{2})B_0 \tag{6.2}$$

where we have

$$[\mathcal{N}, B_0^{\dagger}] = B_0^{\dagger} \qquad [\mathcal{N}, B_0] = -B_0 \qquad [B_0, B_0^{\dagger}] = 2\mathcal{N} \tag{6.3}$$

which identifies it as an sp(2, R) subalgebra whose Casimir operator is then

$$I^2 = I_0(I_0 - 1) + 2I_1 I_{-1} = \mathcal{N}(\mathcal{N} - 1) - B_0^{\dagger} B_0. \tag{6.4}$$

Again from (3.2b) we note that

$$I_1' = \tfrac{1}{2}B_1^{\dagger} \qquad I_0' = \tfrac{1}{2}(\mathcal{N} + J_0) I_{-1}' = \tfrac{1}{2}B_{-1} \tag{6.5}$$

close under commutation as

$$[\mathcal{N} + J_0, B_1^{\dagger}] = 2B_1^{\dagger} \qquad [\mathcal{N} + J_0, B_{-1}] = -2B_{-1}$$
$$[B_{-1}, B_1^{\dagger}] = -2(\mathcal{N} + J_0). \tag{6.6}$$

It also corresponds to an sp(2, R) subalgebra and thus its Casimir operator is

$$I'^2 = \tfrac{1}{4}(\mathcal{N} + J_0)(\mathcal{N} + J_0 - 2) + \tfrac{1}{2}B_1^{\dagger} B_{-1}. \tag{6.7}$$

Finally we note that

$$I_1'' = \tfrac{1}{2}B_{-1}^{\dagger} \qquad I_0'' = \tfrac{1}{2}(\mathcal{N} - J_0) \qquad I_{-1}'' = \tfrac{1}{2}B_1 \tag{6.8}$$

also close under commutation and the Casimir operator is

$$I''^2 = \tfrac{1}{4}(\mathcal{N} - J_0)(\mathcal{N} - J_0 - 2) + \tfrac{1}{2}B_{-1}^{\dagger} B_1. \tag{6.9}$$

As we have already obtained, in (4.10), the effect of the operators $\mathcal{N}$, B_q^+, J_q, B_q on the states $|N, M, \mu, \sigma\rangle$ of (3.6), we can also determine the matrix representation of the Casimir operators of this section, which are given below:

$$\begin{aligned} J^2|N, M, \mu, \sigma\rangle = [&M(M+1)+2\mu(N-\mu+1) \\ &+(N-M+\sigma-\mu)+(s+\sigma+1)(s-\sigma)]|N, M, \mu, \sigma\rangle \\ &+\tfrac{1}{2}(N+M-\sigma-\mu)(N-M+\sigma-\mu)|N, M, \mu+2, \sigma\rangle \\ &+2\mu(\mu-1)|N, M, \mu-2, \sigma\rangle-2\mu|N, M, \mu-1, \sigma+1\rangle \\ &+\tfrac{1}{2}(N-M+\sigma-\mu)(s+\sigma)(\sigma-s-1)|N, M, \mu+1, \sigma-1\rangle \\ &+\mu(s+\sigma)(\sigma-s-1)|N, M, \mu-1, \sigma-1\rangle \\ &-(N+M-\sigma-\mu)|N, M, \mu+1, \sigma+1\rangle \end{aligned} \tag{6.10}$$

$$\begin{aligned} I^2|N, M, \mu, \sigma\rangle = &(N+w-\mu)(N+w-\mu-1)|N, M, \mu, \sigma\rangle \\ &+(N+M-\mu-\sigma)|N, M, \mu+1, \sigma+1\rangle \\ &-\tfrac{1}{2}(N+M-\mu-\sigma)(N-M-\mu+\sigma)|N, M, \mu+2, \sigma\rangle \\ &+\tfrac{1}{2}(s+\sigma)(s-\sigma+1)(N-M-\mu+\sigma)|N, M, \mu+1, \sigma-1\rangle \end{aligned} \tag{6.11}$$

$$\begin{aligned} I'^2|N, M, \mu, \sigma\rangle = &\tfrac{1}{4}(w+\sigma+\mu)(w+\sigma+\mu-2)|N, M, \mu, \sigma\rangle \\ &-\tfrac{1}{2}\mu(\mu-1)|N, M, \mu-2, \sigma\rangle+\tfrac{1}{2}\mu(s+\sigma)(s-\sigma+1)|N, M, \mu-1, \sigma-1\rangle \end{aligned} \tag{6.12}$$

$$\begin{aligned} I''^2|N, M, \mu, \sigma\rangle = &\tfrac{1}{4}(w-\sigma+\mu)(w-\sigma+\mu-2)|N, M, \mu, \sigma\rangle \\ &-\tfrac{1}{2}\mu(\mu-1)|N, M, \mu-2, \sigma\rangle+\mu|N, M, \mu-1, \sigma+1\rangle. \end{aligned} \tag{6.13}$$

In fact we can determine the matrix representation of any Hamiltonian which is a linear combination of these Casimir operators or any arbitrary function of the generators of sp(4, R).

A check on the above matrix elements can be obtained from the fact that the second Casimir operator of sp(4, R) has the form (Castaños *et al* 1985a)

$$\begin{aligned} G_2 &= \mathcal{N}(\mathcal{N}-3)+J^2-\boldsymbol{B}^+\cdot\boldsymbol{B} \\ &= \{J^2+I^2+2I'^2+2I''^2\} \\ &\quad +\{\mathcal{N}-\tfrac{1}{2}(\mathcal{N}+J_0)(\mathcal{N}+J_0-2)-\tfrac{1}{2}(\mathcal{N}-J_0)(\mathcal{N}-J_0-2)\} \end{aligned} \tag{6.14}$$

where the right-hand side is obtained from (6.4), (6.7) and (6.9). The eigenvalue of G_2 can be determined by applying the operator to the lowest weight state $|w, s, -s\rangle$ that satisfies $B_q|w, s, -s\rangle=0$. Thus we have

$$G_2|N, M, \mu, \sigma\rangle = [w(w-3)+s(s+1)]|N, M, \mu, \sigma\rangle \tag{6.15}$$

and from (6.14) we see that

$$\begin{aligned} &\{J^2+I^2+2I'^2+2I''^2\}|N, M, \mu, \sigma\rangle \\ &\quad = [w(w-3)+s(s+1)+(N+w)^2+M^2]|N, M, \mu, \sigma\rangle \end{aligned} \tag{6.16}$$

which also follows from (6.10)-(6.13).

7. Conclusion

The analysis carried out in this paper concerns the states that form a basis for irreps in the positive discrete series of sp(4, R), and on the determination of the overlaps of these states and the matrix representation of the generators with respect to this basis. The title of this paper, however, speaks of symplectic algebras in general and the question is whether the procedures presented here can be applied to sp($2d$, R) where d is any integer.

The answer is yes, as we can divide the generators of sp($2d$, R) into three sets of raising, weight and lowering type as in (3.3), define the lowest weight state as in (3.4) and our full basis by applying the raising generators to the lowest weight state as in (3.6). It is also possible to find a Dyson-type boson realisation for the sp($2d$) generators (Moshinsky 1984a) corresponding to (4.2) which will now contain generators of su(d) as well, like (4.2) contained the S_q of su(2). In turn these su(d) generators have a Dyson boson realisation as in (2.21) for su(2). We can then express our basis purely in terms of creation operators acting on a vacuum state and the generators as differential operators with respect to the creation operators as indicated in (4.9), thus allowing us to determine the matrix representation as in (4.10). Furthermore one can define coherent states for sp($2d$, R) (Quesne 1986) and from them derive the overlaps of the states of our basis in analogy with what was done in § 5.

We plan to carry out this programme for sp(6, R) in a forthcoming publication.

Appendix. Determination of the overlap (5.2)

From the discussion in § 5 we see that to get the overlap (5.2) we only need to expand

$$\mathcal{M}^{2s}(\Delta)^{-w-s} \tag{A1}$$

in powers of y', $\bar{y}$ and of the components of the vectors z', $\bar{z}$. We start with $\mathcal{M}^{2s}$ of (5.8b) and, developing first in powers of y', $\bar{y}$, we have

$$\mathcal{M}^{2s}=\sum_{\sigma,\sigma'} F^{s}_{\sigma,\sigma'}(h,k,l,m)y'^{s+\sigma'}\bar{y}^{s+\sigma} \tag{A2}$$

where

$$F^{s}_{\sigma,\sigma'}(h,k,l,m)=\sum_{r}\frac{(2s)!}{(2s-r)!(r-s-\sigma)!(r-s-\sigma')!(2s+\sigma+\sigma'-r)!}\times(h)^{2s-r}(k)^{r-s-\sigma}(l)^{r-s-\sigma'}(m)^{2s+\sigma+\sigma'-r}. \tag{A3}$$

As h, k, l, m are given in turn by (5.9) we expand them by the binomial theorem in powers of z'_1, z'_0, z'_{-1}, $\bar{z}^1$, $\bar{z}^0$, $\bar{z}^{-1}$ and thus finally get

$$\mathcal{M}^{2s}=\sum_{\bar{\lambda}'\bar{\mu}'\bar{\nu}'\sigma'}\sum_{\bar{\lambda}\bar{\mu}\bar{\nu}\sigma} C(\bar{\lambda}',\bar{\mu}',\bar{\nu}',\sigma';\bar{\lambda},\bar{\mu},\bar{\nu},\sigma)\times(z'_1)^{\bar{\lambda}'}(z'_0)^{\bar{\mu}'}(z'_{-1})^{\bar{\nu}'}y'^{s+\sigma'}(\bar{z}^1)^{\bar{\lambda}}(\bar{z}^0)^{\bar{\mu}}(\bar{z}^{-1})^{\bar{\nu}}\bar{y}^{s+\sigma} \tag{A4}$$

where

$$C(\bar{\lambda}',\bar{\mu}',\bar{\nu}',\sigma';\bar{\lambda},\bar{\mu},\bar{\nu},\sigma)=\sum_{r}\frac{(-1)^{\bar{\lambda}'+\bar{\nu}'+\bar{\mu}'}2^{\bar{\lambda}'-s-\sigma+\bar{\nu}}(3s+\sigma-r-\bar{\lambda}'-\bar{\nu})!(2s)!}{(\bar{\mu}'+\bar{\nu}'-\bar{\nu}+s+\sigma-r)!(2s-\bar{\lambda}'-\bar{\mu}'-\bar{\nu}')!}D_r \tag{A5}$$

and in which

$$D_r = \sum_d [(s+\bar{\nu}'-\bar{\nu}-\bar{\lambda}'-\sigma'-d)!(\bar{\lambda}'+s+\sigma'-r+\bar{\nu}-\bar{\nu}'+d) \times(\bar{\nu}'-d)!(2s+\sigma+\sigma'-r-\bar{\nu}'+d)! \times(r-s-\sigma-d)!d!(\bar{\nu}-\bar{\nu}'+d)!(r-s-\sigma+\bar{\nu}'+\bar{\nu}-d)!]^{-1}. \quad \text{(A6)}$$

On the other hand, we showed in equation (A7) of Castaños *et al* (1986) that

$$\Delta^{-(w+s)} = \sum_{\gamma\beta\alpha} \frac{(\gamma+2w+2s-\beta-3)!!(\gamma+2w+2s+\beta-2)!!}{(2w+2s-2)!!(2w+2s-3)!!} P_{\gamma\beta\alpha}(z')P_{\gamma\beta\alpha}(z) \quad \text{(A7)}$$

where $P_{\gamma\beta\alpha}(z)$ are the polynomials

$$P_{\gamma\beta\alpha}(z) = \sum_\tau \frac{(z_1)^{(\gamma-\tau+\alpha)/2}(z_0)^\tau(z_{-1})^{(\gamma-\tau-\alpha)/2}}{\{[(\gamma-\tau+\alpha)/2]!\tau![(\gamma-\tau-\alpha)/2]!\}^{1/2}}(\gamma\tau\alpha\,|\,\gamma\beta\alpha) \quad \text{(A8)}$$

and the bracket has the form (Chacón and de Llano 1963)

$$(\gamma\tau\alpha\,|\,\gamma\beta\alpha) = (-1)^{(\gamma-\beta)/2}2^{(\gamma-\tau)/2-\beta}\{(2\beta+1)(\beta-\alpha)!\tau! \times[(\gamma+\alpha-\tau)/2]!\}^{1/2}\{(\gamma+\beta+1)!!(\gamma-\beta)!!(\beta+\alpha)![(\gamma-\alpha-\tau)/2]!\}^{-1/2} \times\sum_{k=0}^{\beta}\frac{(-1)^k[(\gamma-\beta)/2+k]!(2\beta-2k)!}{k!(\beta-k)!(\beta-2k-\alpha)![(\alpha+\tau-\beta)/2+k]!} \quad \text{(A9)}$$

From (A4) and (A7) we can immediately obtain the expansion of $\mathcal{M}^{2s}\Delta^{-(w+s)}$ in powers of y', $\bar{y}$ and the components of z', $\bar{z}$. The coefficients of these products of powers give the overlap as indicated in (5.10).

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